The Brownian Bridge for Space-Time Interpolation

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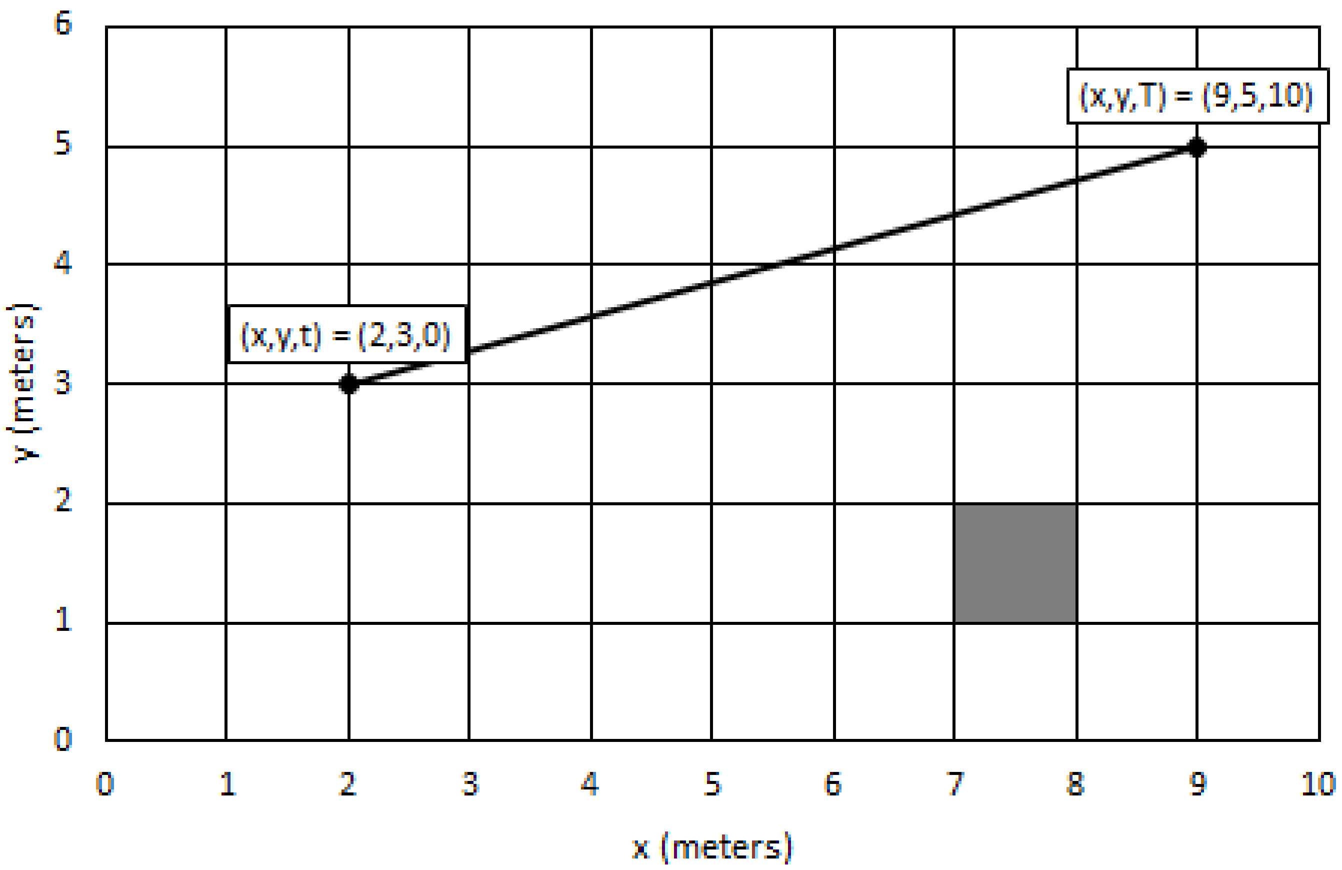
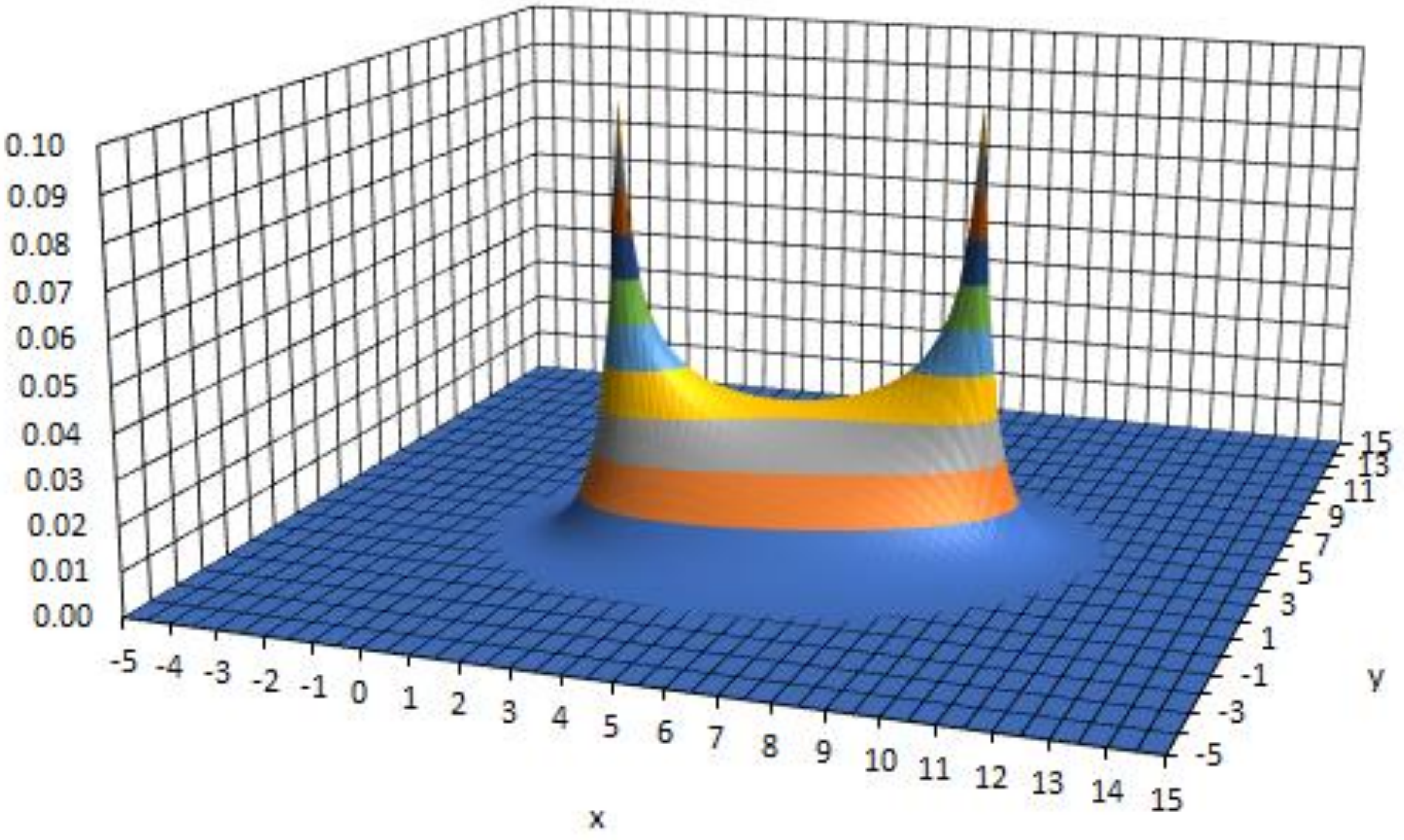
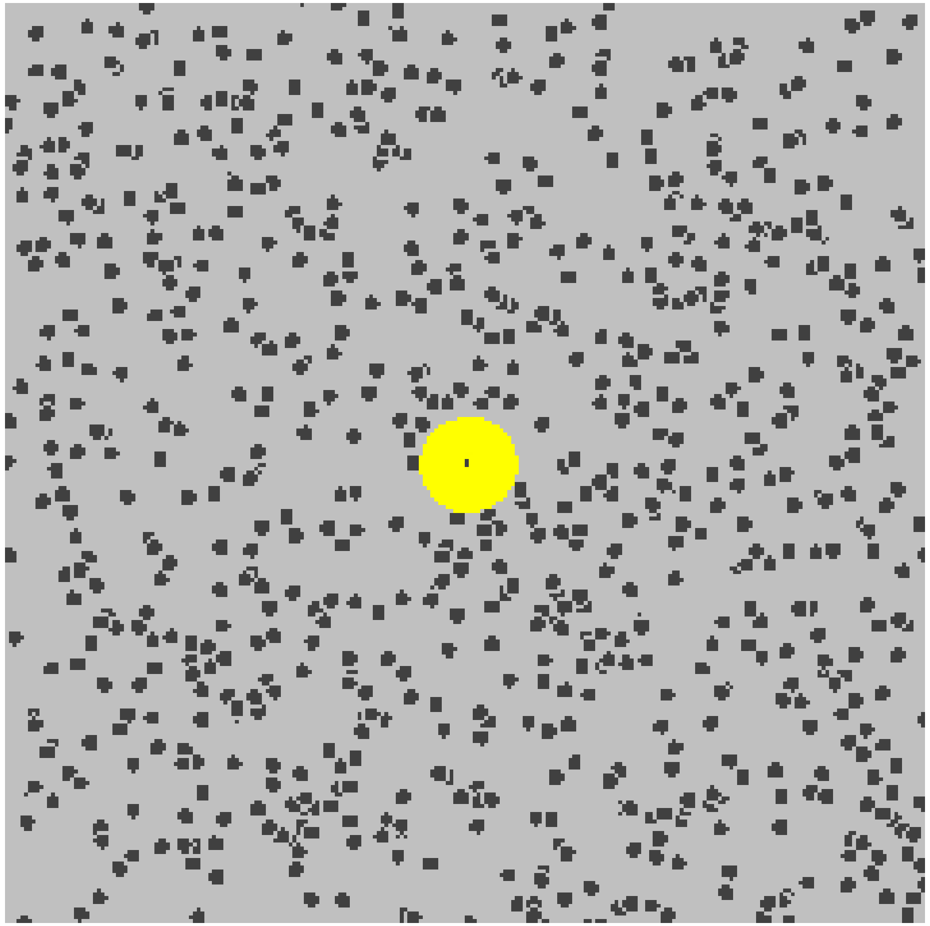
Where did the person go between measurements?

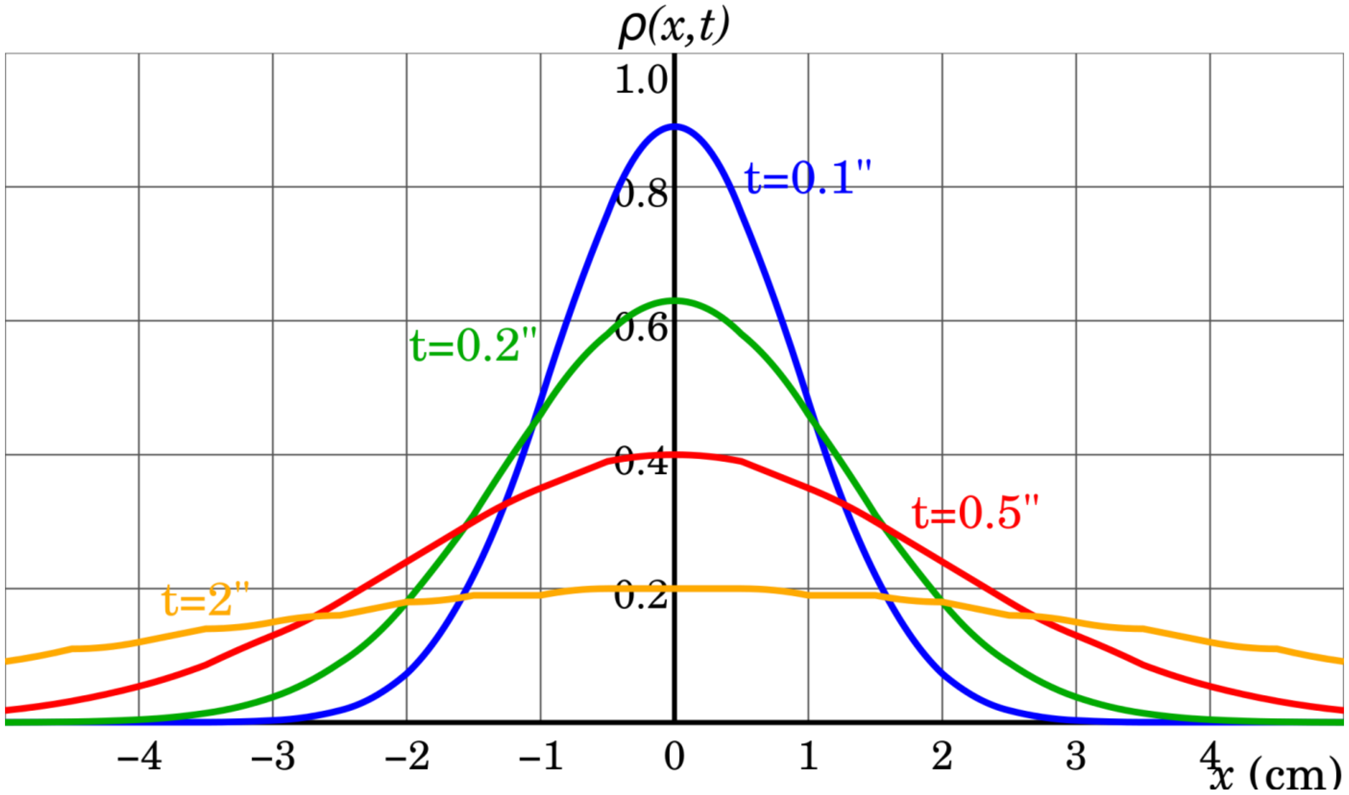


Brownian Bridge with Time Integrated Out

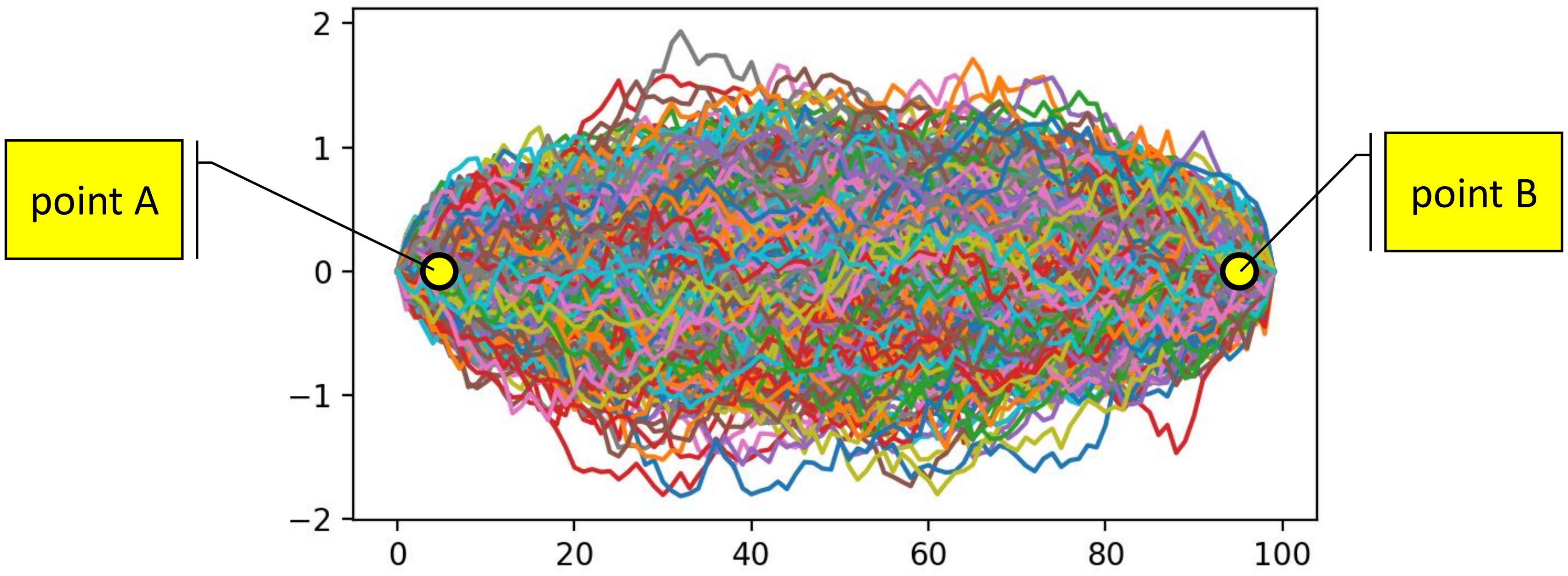


Brownian Motion





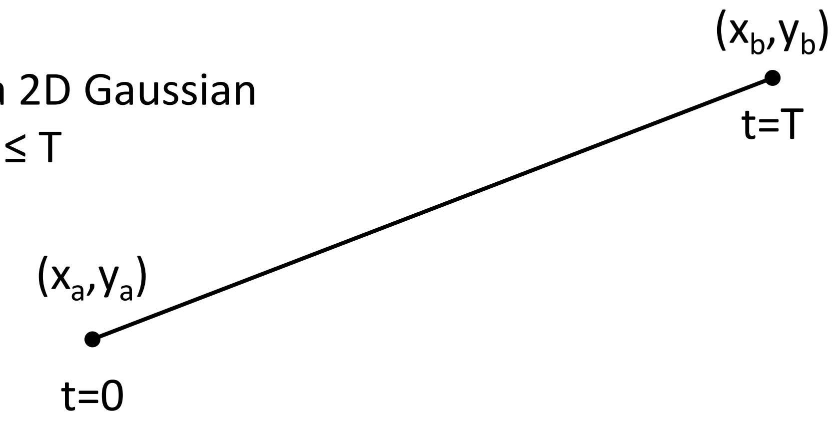
Brownian Bridge



$$f(\mathbf{x}|t) = \frac{1}{2\pi\sqrt{|\Sigma(t)|}} e^{-\frac{1}{2}\left[\left(\mathbf{x}-\mu(t)\right)^T \Sigma^{-1}(t)\left(\mathbf{x}-\mu(t)\right)\right]}$$

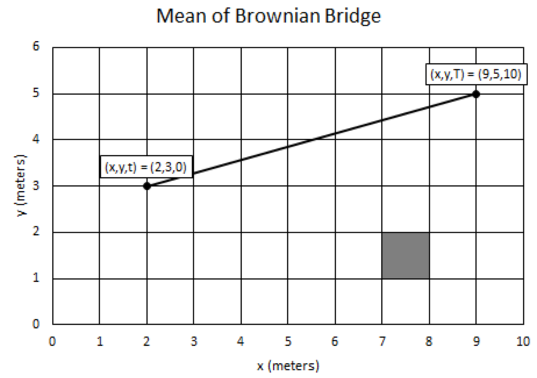
$$= \frac{1}{2\pi\sigma^2(t)} e^{-\frac{1}{2\sigma^2(t)}\|\mathbf{x}-\mu(t)\|^2} \text{ for } 0 < t < T$$

Brownian bridge is a 2D Gaussian at each time T, 0 ≤ t ≤ T



$$\mu(t) = \mathbf{x}_a + \frac{t}{T}(\mathbf{x}_b - \mathbf{x}_a) \text{ for } 0 < t < T$$

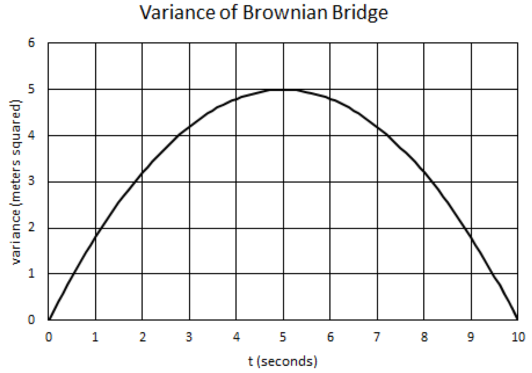
Mean of each Gaussian is linear interpolation from point **a** to point **b**.



$$\Sigma(t) = \begin{bmatrix} \sigma^2(t) & 0 \\ 0 & \sigma^2(t) \end{bmatrix} \text{ for } 0 < t < T$$

$$\sigma^2(t) = \frac{t(T-t)}{T}\sigma_m^2 \text{ for } 0 < t < T$$

Variance of each Gaussian is parabola from t = 0 to t = T.



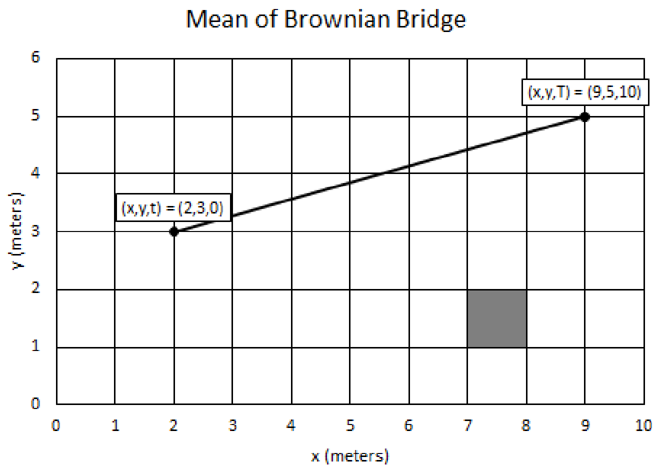
Integrate for Probability

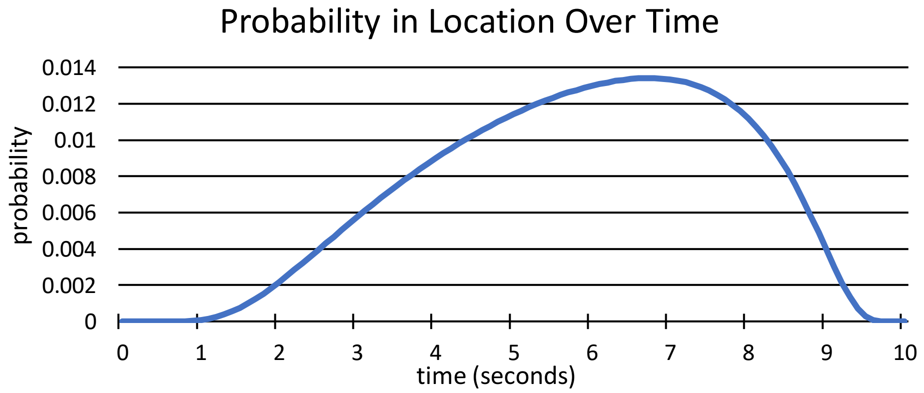
$$P(\mathbf{x} \in \mathcal{R}|t) = \int\int_{\mathcal{R}} f(\mathbf{x}|t)\, d\mathbf{x}$$

$$= \int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x, y|t)\, dx\, dy$$

$$= \frac{1}{4}\left[\left(\operatorname{erf}\left(\frac{x_2 - \mu_x(t)}{\sqrt{2}\sigma(t)}\right) - \operatorname{erf}\left(\frac{x_1 - \mu_x(t)}{\sqrt{2}\sigma(t)}\right)\right)\left(\operatorname{erf}\left(\frac{y_2 - \mu_y(t)}{\sqrt{2}\sigma(t)}\right) - \operatorname{erf}\left(\frac{y_1 - \mu_y(t)}{\sqrt{2}\sigma(t)}\right)\right)\right]$$

Probability of being in a certain region as a function of time

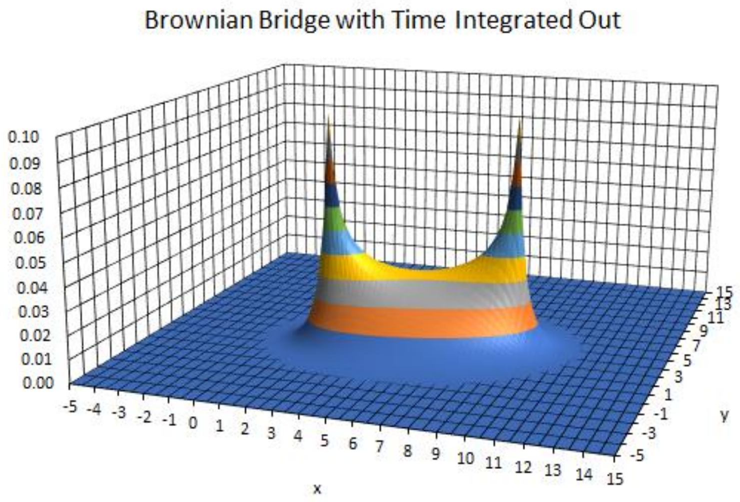




Integrate for Probability

$$f(\mathbf{x}) = \frac{1}{T}\int_0^T f(\mathbf{x}|t)dt$$

Expected location
aggregated over 0 ≤ t ≤ T



$$f(x,y) = \frac{1}{\pi\sigma_m^2 T}\exp\Big(-\frac{1}{\sigma_m^2 T}(-x'b_r + x'^2 + y'^2)\Big)K_0\Big(\frac{1}{\sigma_m^2 T}\sqrt{x'^2+y'^2}\sqrt{b_r^2 - 2x'b_r + x'^2 + y'^2}\Big)$$

Useful for computing “home range” of animals

Steffie Van Nieuland, Jan M Baetens, Hans De Meyer, and Bernard De Baets. An analytical description of the time-integrated brownian bridge. *Computational and Applied Mathematics*, 36(1):627–645, 2017.

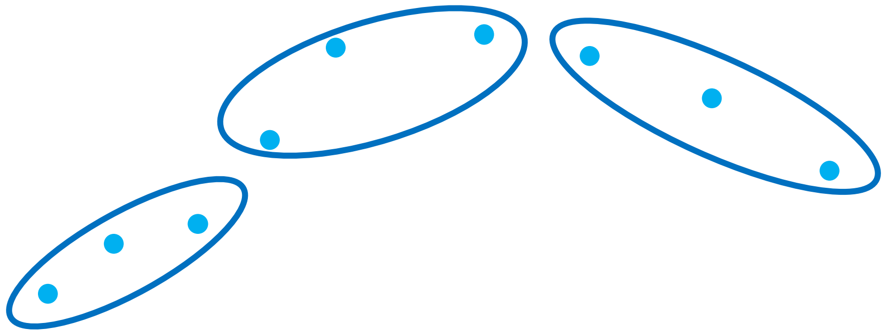
Fit to Data

$$f(\mathbf{x}|t) = \frac{1}{2\pi\sqrt{|\Sigma(t)|}} e^{-\frac{1}{2}\left[\left(\mathbf{x}-\mu(t)\right)^T \Sigma^{-1}(t)\left(\mathbf{x}-\mu(t)\right)\right]}$$

$$= \frac{1}{2\pi\sigma^2(t)} e^{-\frac{1}{2\sigma^2(t)}\|\mathbf{x}-\mu(t)\|^2} \text{ for } 0 < t < T$$

$$\sigma^2(t) = \frac{t(T-t)}{T}\sigma_m^2 \text{ for } 0 < t < T$$

“diffusion coefficient”, units length^2/time



Simple fit using triples of points and maximum likelihood

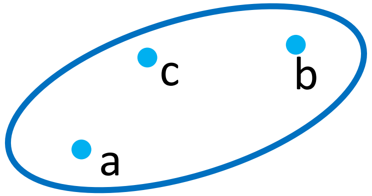
Fit to Data – Simple Example

<u>Triple of points</u>

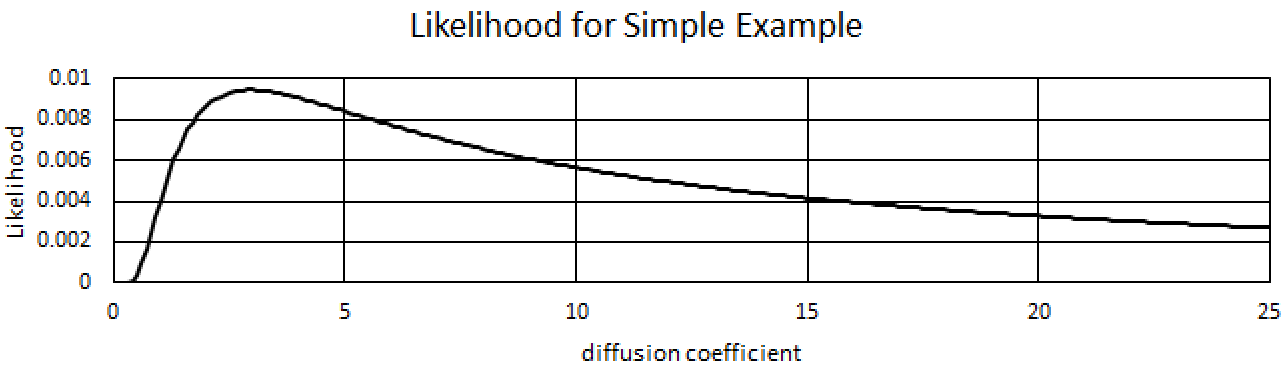
$(x_a, y_a, t_a) = (2,3,0)$

$(x_c, y_c, t_c) = (6,1,7)$

$(x_b, y_b, t_b) = (9,5,10)$

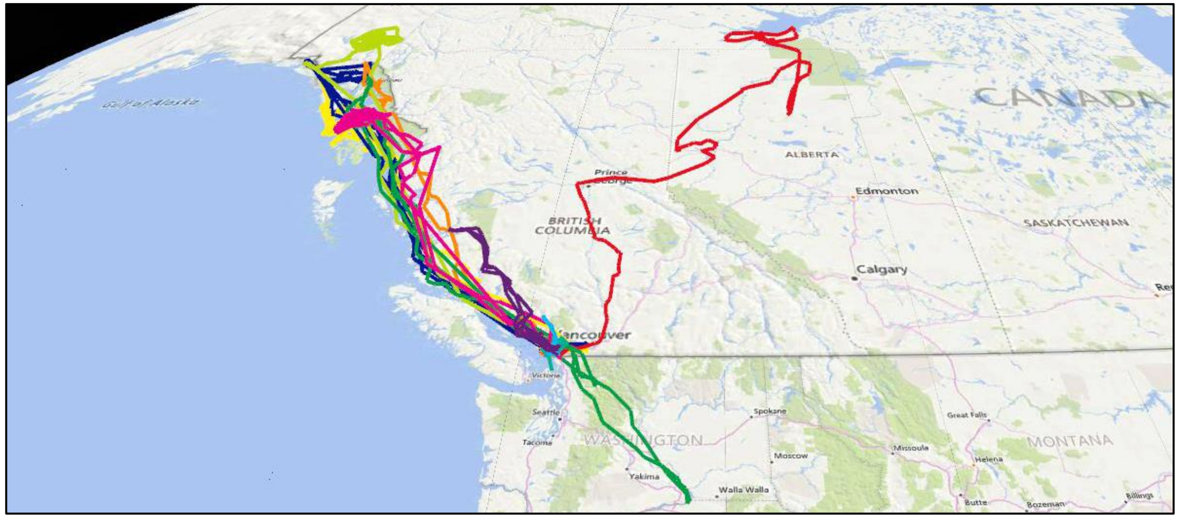


$$
\begin{aligned}
i &= 1 \\
T_i &= 10 \\
t_i &= 7 - 0 = 7 \\
\mu(t_i) &= (2,3)^T + \frac{7}{10}\left((9,5)^T - (2,3)^T\right) \\
&= (6.9, 4.4)^T \\
\sigma^2(t_i) &= \frac{7(10-7)}{10}\sigma_m^2 \\
&= 2.1\sigma_m^2 \\
\|\mathbf{x_{i,c}} - \mu(t_i)\|^2 &= \|(6,1)^T - (6.9,4.4)^T\|^2 \\
&= \|(-0.9,-3.4)\|^2 \\
&= 12.37 \\
L &= L_i \\
&= \frac{1}{4.2\pi\sigma_m^2} e^{-\frac{12.37}{4.2\sigma_m^2}}
\end{aligned}
$$

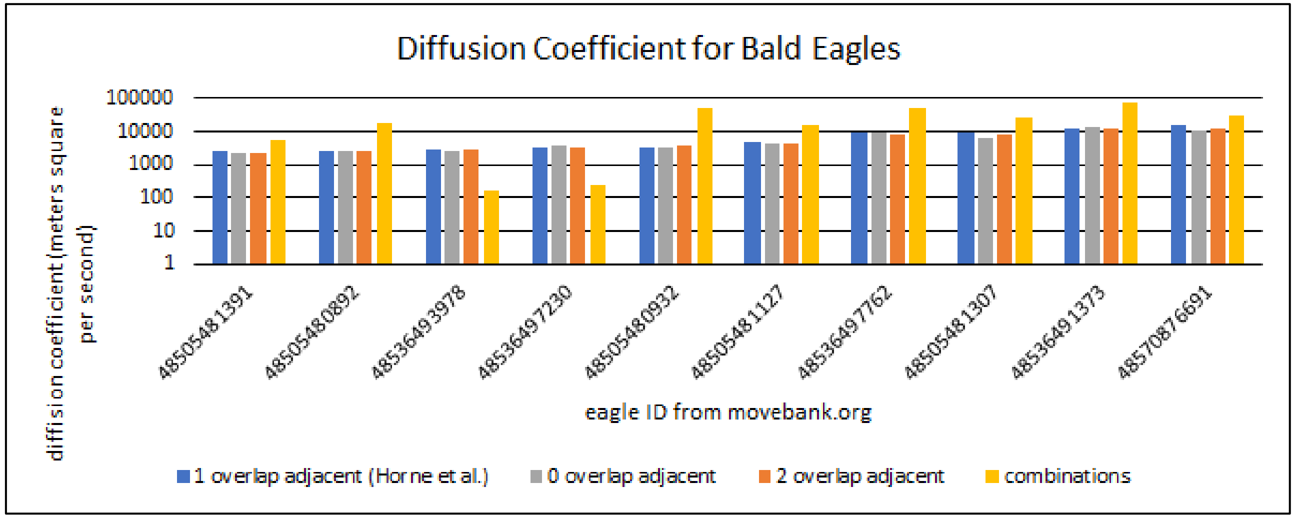


$$\sigma_m^2 = 2.945$$

Fit to Data – Bald Eagles Example



Data from 10 bald eagles via movebank.org

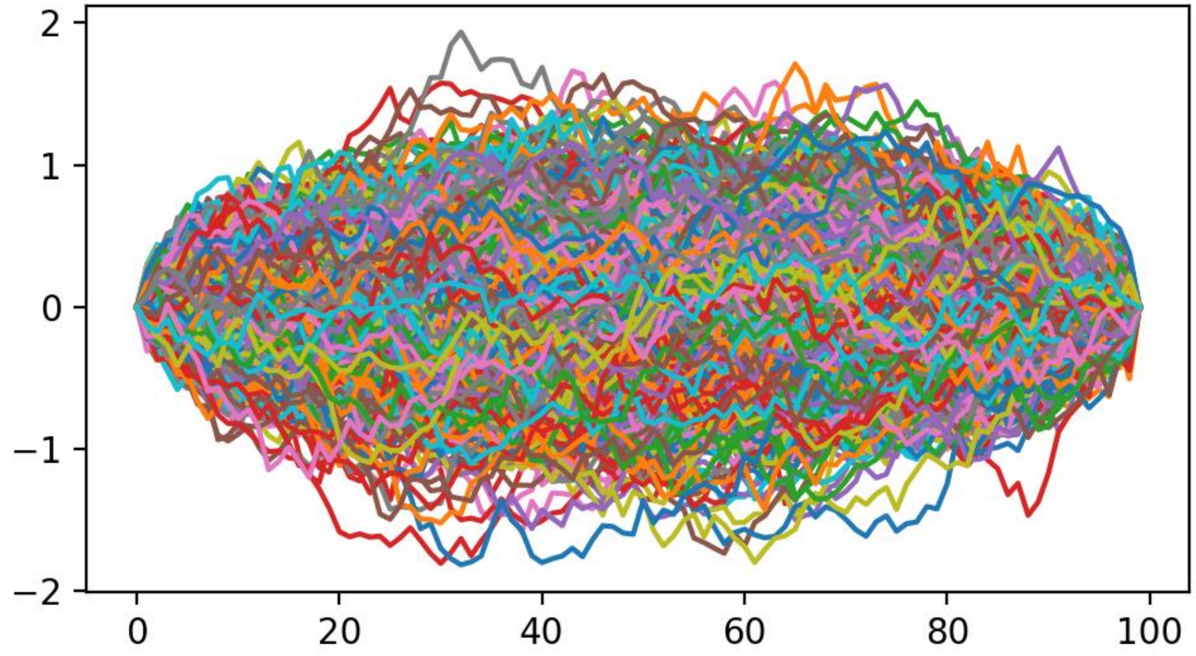


Computed diffusion coefficients (Brownian bridge)

Myles Lamont. Bald Eagle (Haliaeetus leucocephalus) in British Columbia. https://www.movebank.org/cms/webapp?gwt_fragment=page=studies,path=study430263960, 2018.

Summary

- Introduce Brownian bridge
- Integrate
  - Probability over time of occupying region
  - Probability over location during travel from **a** to **b**
- Fit Brownian bridge to data (diffusion coefficient)
  - General approach
  - Examples
    - Simple 3-point example
    - Data from eagles



Corrections in Response to Reviews

- Equation (9) is only instantaneous in time.
  - “For instance, we may want to know if a person visited a certain store or may have been exposed at a certain virus hotspot at given time instant.”
  - “Note that Equation 9 gives an instantaneous probability at time t of the object being inside the given rectangle. It does not give a probability of visiting the region over some time interval.”
- Humans and animals don’t move as randomly as a Brownian bridge.
  - “Humans and animals likely move with more intention than the Brownian bridge implies, yet the model can be useful for acknowledging and representing uncertainty about what happens between measurements.”
- Other work
  - “Alternatives to this probabilistic representation reason about the maximum speed of an object traversing between two points, e.g. [1,2]. In [3], the authors present a thorough discussion of modeling and querying about objects whose uncertain location is represented by Markov chains through discrete time and space.”
- The legends in Figure 2 are barely readable.
  - Increased overall size of subfigures.